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Large Number Hypothesis and The Matter-Dominated Universe[1]

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Dirac's large number hypothesis (LNH), in the form $G/G_0 = HH_0^{-1}$, is applied to the matter-dominated cosmological era, using the framework of the scale covariant theory (Canuto et al., 1977). We obtain explicit expressions for R and β_a as functions of R_E, where R and R_E are the scale factors of the cosmological Robertson–Walker metric, expressed in atomic and gravitational units, respectively, and β_a is the ratio between the rates of gravitational and atomic clocks. The parameters in these expressions are $\bar{q}_0$, the deceleration parameter in gravitational units, and $\dot{\beta}_a(t_0)H_0^{-1}$ where $\dot{\beta}_a(t_0)$ is the present epoch value of the derivative of β_a with respect to atomic time. We find that a necessary condition for the LNH to be compatible with a Robertson–Walker model is that $\dot{\beta}_a(t_0)H_0^{-1} \geqslant 1/2$. The only experimental values for $\dot{\beta}_a(t_0)$ available at present are those based on the lengthening of the Moon's period of revolution around the Earth, suggesting $0.86 \geqslant \dot{\beta}_a(t_0)H_0^{-1} \geqslant 0.21$; the more promising technique of radar ranging to the inner planets has not yet produced a value for $\dot{\beta}_a(t_0)$. Using the lunar data, it follows that $0 \leqslant \bar{q}_0 \lesssim 0.42$ corresponding to an open universe ($k = -1$). Closed models ($k = 1, \bar{q}_0 > 1/2$) are not compatible with the LNH since the required values of $\dot{\beta}_a(t_0)H_0^{-1}$ are more than an order of magnitude above the observational upper limit.

1. INTRODUCTION

Dirac (1937, 1973, 1974) has proposed an interesting and inspiring interpretation for the numerical relation between the two dimensionless large numbers

$$\frac{e^2}{G_0 m^2} = \lambda H_0^{-1} \frac{mc^3}{e^2} \approx 10^{39} \tag{1}$$

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Here G_0 and H_0 are the present epoch values of the gravitational coupling and the Hubble parameter, respectively, e is the electron charge, m is a mass of a typical elementary particle (e.g., the mass of a pion), and λ is a dimensionless constant of order unity. The left-hand side gives the ratio of the electric to the gravitational force acting between two particles of mass m and charge e. The right-hand side is a ratio of the cosmological Hubble time (H_0^{-1}) to an atomic time scale, i.e., it gives the Hubble time as expressed in atomic time units. Dirac assumed, in what is known as the large number hypothesis (LNH), that (1) is not merely a numerical coincidence but rather a general relation which should hold for any cosmological epoch, namely,

$$\frac{e^2}{Gm^2} = \lambda H^{-1} \frac{mc^3}{e^2} \tag{2}$$

Since the universe is nonstatic, (2) implies that the gravitational coupling, when expressed in atomic units, varies with the cosmological epoch. From (1) and (2) one gets

$$G(t)/G_0 = H_0^{-1} H(t) \tag{3}$$

By taking $H(t) \sim 1/t$, Dirac has obtained

$$\frac{G(t)}{G_0} = \frac{t_0}{t} \tag{4}$$

a form which has been commonly used in the literature.

In the present work we study the implications of the LNH, in the form (3), on the cosmological models describing the matter-dominated era. As stressed by Dirac (1978), relations (3) [or (4)] are expected to be modified in the early radiation-dominated universe. The analysis is carried out in the framework of the scale covariant theory (Canuto et al., 1977), the advent of which was strongly motivated by Dirac's ideas.

The scale covariant theory is a theoretical framework which allows for a possible violation of the strong equivalence principle (SEP), or equivalently a variation of G with cosmic time, through the existence in nature of two distinct fundamental systems of units: *gravitational or Einstein units* based on gravity clocks (e.g., a planet orbiting a star) and *atomic units* based on atomic clocks. Einstein field equations, as the correct description of macroscopic gravity, apply in gravitational units only. The proper length differences connecting two space-time events, as measured by the two different clocks, are related by

$$\Delta s_E = \beta_a(t)\, \Delta s_a \tag{5}$$

where Δs_E is in gravitational units, Δs_a is in atomic units, and where $\beta_a(t)$ is a dimensionless function of the cosmological epoch, with its present value $\beta_a(t_0)$ normalized to unity. A nonconstant β_a gives rise to a variable gravitational coupling in atomic units through

$$G_a = \beta_a^{-g} G_E, \qquad G_E, g = \text{const} \tag{6}$$

where g is a constant parameter and G_E is the constant gravitational coupling in gravitational units. It has been shown recently (Canuto and Goldman, 1982) that in order that the two different clocks be constructed consistently from their dynamical equations, the value

$$g = 2 \tag{7}$$

is required.

Note that when tracking the same phenomenon using both gravitational and atomic clocks, one measures directly $\dot{\beta}_a$ and not $\dot{G}_a$, which is derived quantity. Such measurements of the lengthening of the moon's period of revolution around the earth do suggest a value (Van Flandern, 1981)

$$\dot{\beta}_a(t_0) = (3.2 \pm 1.1) \times 10^{-11}\,\text{yr}^{-1} \tag{8}$$

where $\beta_a(t_0)$ is normalized to unity.

There remain uncertainties related to the analysis of the tidal forces in the Earth–Moon system. A more reliable value may be obtained by using radar ranging measurements of the inner tideless planets, but at present the estimated errors are still of the order of the effect itself, so only an upper limit $|\dot{\beta}_a(t_0)| \lesssim 10^{-10}\,\text{yr}^{-1}$ is available (Shapiro, 1981).

2. THE MODEL

The cosmological Robertson–Walker metric, in gravitational units, is given by (Weinberg, 1972)

$$ds_E^2 = dt_E^2 - R_E^2(t_E)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right) \tag{9}$$

where $k = 0, \pm 1$.

Using (5), the metric in atomic units is found to be

$$ds_a^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right) \tag{10}$$

with

$$dt = \beta_a^{-1} dt_E, \qquad R = R_E \beta_a^{-1} \tag{11}$$

In gravitational units, Einstein equations do apply, so that for the matter-dominated era one has (the cosmological constant set equal to zero) (Weinberg, 1972),

$$R'_E = \bar{H}_0 R_0 (1 - 2\bar{q}_0 + 2\bar{q}_0 R_0 / R_E)^{1/2} \tag{12}$$

where a prime denotes a derivative with respect to the gravitational time t_E and $\bar{H}_0$ and $\bar{q}_0$ are the Hubble constant and deceleration parameter in gravitational units, given by

$$\bar{H}_0 = \left(\frac{R'_E}{R_E} \right)_0, \qquad \bar{q}_0 = -\left(\frac{R''_E}{R_E} \right)_0 \bar{H}_0^{-2} \tag{13}$$

and

$$R_0 = R_E(t_0) = \beta_a(t_0) R(t_0) = R(t_0) \tag{14}$$

Values of $\bar{q}_0 > 1/2$ correspond to a spatially closed universe ($k = 1$), in which R_E is bound

$$R_E \leqslant R_0 \cdot \frac{2\bar{q}_0}{2\bar{q}_0 - 1}$$

For $\bar{q}_0 = 1/2$ the universe is spatially flat ($k = 0$) while $0 \leqslant \bar{q}_0 < 1/2$ correspond to a spatially hyperbolic open universe ($k = -1$). In both cases $k = 0, -1$, R_E is unbound.

The LNH relation (3) becomes, using (6) and (7),

$$\beta_a^{-2} = H_0^{-1} H = H_0^{-1} \dot{R} / R \tag{15}$$

where a dot represents a derivative with respect to the atomic time t. From (11) we get

$$\frac{\dot{R}}{R} = \frac{\dot{R}_E}{R_E} - \frac{\dot{\beta}_a}{\beta_a} = \frac{R'_E}{R_E} \beta_a - \beta'_a \tag{16}$$

Combining (12), (15), and (16) we obtain

$$\frac{d}{dx}\left(\frac{\beta_a^3}{x^3}\right) = -3\frac{H_0}{\bar{H}_0}x^{-3}\left(1-2\bar{q}_0+\frac{2\bar{q}_0}{x}\right)^{-1/2} \tag{17}$$

where $x = R_E/R_0$. Equation (17) can be integrated to yield

$$\left(\frac{\beta_a}{x}\right)^3 = 1-\frac{H_0}{\bar{H}_0\bar{q}_0^2}\left[\left(1-2\bar{q}_0+\frac{2\bar{q}_0}{x}\right)^{1/2}\left(1-2\bar{q}_0-\frac{\bar{q}_0}{x}\right)+3\bar{q}_0-1\right] \tag{18}$$

where the normalization $\beta_a(x=1)=1$ has been imposed.

$\bar{H}_0$ can be expressed through H_0 and $\dot{\beta}_a(t_0)$ by setting $t=t_0$ in equation (16). One gets

$$\bar{H}_0 = H_0 + \dot{\beta}_a(t_0) \tag{19}$$

which when substituted in (18) gives finally

$$\frac{\beta_a^3}{x^3} = \left(\frac{R_0}{R}\right)^3$$

$$= 1 - \frac{1}{1+\dot{\beta}_a(t_0)H_0^{-1}}\cdot\frac{1}{\bar{q}_0^2}\left[\left(1-2\bar{q}_0+\frac{2\bar{q}_0}{x}\right)^{1/2}\left(1-2\bar{q}_0-\frac{\bar{q}_0}{x}\right)+3\bar{q}_0-1\right] \tag{20}$$

3. DISCUSSION

Equation (20) gives explicitly R and β_a as functions of x (or R_E) with $\bar{q}_0$ and $\dot{\beta}_a(t_0)H_0^{-1}$ serving as parameters. These relations are useful in the analysis of cosmological data, which may in turn yield limits on the parameters involved.

Quite strong constraints on these parameters may be derived by requiring that in the matter-dominated era β_a as given by (20) should be positive [as required from (5) and (11)]. Consider therefore the following categories of cosmological models:

(a) Closed Models ($\bar{q}_0 > 1/2$; $k=1$). In this case

$$x \leqslant x_{\max} = \frac{2\bar{q}_0}{2\bar{q}_0 - 1} > 1$$

During the contracting phase the sign of the first term in the brackets of (20) gets inverted (since $R'_E < 0$ in this phase). For $x = 0.1$ in the contracting phase the universe is, no doubt, matter dominated and therefore (20) is applicable, giving rise to

$$(10\beta_a)^3 = 1 - \frac{1}{1+\dot{\beta}_a(t_0)H_0^{-1}}\frac{1}{\bar{q}_0^2}\left[3\bar{q}_0 - 1 + (18\bar{q}_0+1)^{1/2}(12\bar{q}_0 - 1)\right] \quad (21)$$

Taking $\bar{q}_0 \leqslant 10$, we find that in order that β_a be positive one must have

$$\dot{\beta}_a(t_0)H_0^{-1} > 15.3 \quad (22)$$

Considerably larger values arise for smaller values of $\bar{q}_0$ [e.g., $\bar{q}_0 = 1$ leads to $\beta_a(t_0)H_0^{-1} > 49$].

(b) Open Models ($0 \leqslant \bar{q}_0 \leqslant 1/2$; $k = 0, -1$). In these models x is unbound and equation (20) is therefore applicable for values of x up to infinity. Equation (17) implies that

$$\frac{d}{dx}\left(\frac{\beta_a^3}{x^3}\right) < 0$$

so that $(\beta_a/x)^3$ is minimal for $x \to \infty$. In order for β_a to be positive it is therefore required that

$$\lim_{x\to\infty} (\beta/x)^3 \geqslant 0 \quad (23)$$

In case that the inequality sign in (23) holds, R tends to a finite value when R_E becomes infinite, and the matter density, in atomic units, which is given for $g = 2$ by $\rho_m \sim \beta_a R^{-3}$, (Canuto et al., 1977) becomes infinite. In order to avoid such a singular situation only the case in which the equality sign holds in (23) will be considered. Using (20) one gets,

$$\dot{\beta}_a(t_0)H_0^{-1} = (1/\bar{q}_0^2)\left[(1-2\bar{q}_0)^{3/2} - 1 + 3\bar{q}_0 - \bar{q}_0^2\right] \quad (24)$$

For $\bar{q}_0 = 1/2$ ($k = 0$) one gets $\dot{\beta}_a(t_0)H_0^{-1} \geqslant 1$ while for $\bar{q}_0 = 0$ ($k = -1$) $\dot{\beta}_a(t_0)H_0^{-1} \geqslant 1/2$ is required, with intermediate values resulting for intermediate values of $\bar{q}_0$. It follows therefore that

$$\dot{\beta}_a(t_0)H_0^{-1} \geqslant 1/2 \quad (25)$$

is required in order that *the LNH be consistent with any Robertson–Walker metric at all.*

Taking $H_0 = (50\text{–}100)\ \text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$ and the value of $\dot{\beta}_a(t_0)$ suggested by the lunar data, as given in equation (8), we find

$$0.86 \geqslant \dot{\beta}_a(t_0) H_0^{-1} \geqslant 0.21 \tag{26}$$

which when substituted into (24) yields

$$0 \leqslant \bar{q}_0 \lesssim 0.42 \tag{27}$$

Thus, an open universe with $k = -1$ is suggested. Such small values for $\bar{q}_0$ were obtained also by an approach consistent with (4) (Canuto, Hsieh, and Owen, 1980) and were also found to agree best with the (m, z) diagram (Canuto and Hsieh, 1980); they seem also to be required in order to produce the correct deuterium abundance (Yang et al., 1979). It is hoped that future results from the radar ranging measurements of the inner planets will provide a better estimate of $\dot{\beta}_a(t_0)$ to be compared with (24) and (25).

The values of $\dot{\beta}_a(t_0) H_0^{-1}$ required in the $k = +1$ models [see equation (22)] are well above the experimental uncertainties; thus the LNH is not consistent with a closed universe.

4. CONCLUSIONS

Application of the LNH to the matter-dominated cosmological era yields explicit expressions for β_a and R as functions of R_E, with $\bar{q}_0$ and $\dot{\beta}_a(t_0) H_0^{-1}$ serving as parameters. By definition $\beta_a > 0$, a condition which when imposed on the above-mentioned solutions leads to constraints involving $\dot{\beta}_a(t_0) H_0^{-1}$ and $\bar{q}_0$.

We find the following:

(a) $\dot{\beta}_a(t_0) H_0^{-1} > 1/2$ is required for the LNH to be consistent with any Robertson–Walker cosmology at all.

(b) Closed models $(k = 1, \bar{q}_0 > 1/2)$ are not consistent with the LNH since the required values of $\dot{\beta}_a(t_0) H_0^{-1}$ are more than an order of magnitude larger than the experimental upper limit.

(c) The available experimental values for $\dot{\beta}_a(t_0)$ based on lunar data are consistent with open $(k = -1)$ models characterized by $0 \leqslant \bar{q}_0 \leqslant 0.42$.

It is our hope that future results from the planet radar ranging measurements will yield a more accurate value for $\dot{\beta}_a(t_0)$ to be compared with requirement (a) above.

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